Isometries for blocks of finite groups afforded by the Glauberman correspondences (Cohomology of Finite Groups and Related Topics)

Watanabe, Atumi

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渡辺 アツミ
Atumi Watanabe

Department of Mathematics, Faculty of Science
Kumamoto University, Kumamoto, Japan

§ 1. Perfect isometry

In [W1] we showed that the Glauberman correspondences for irreducible characters of finite groups give perfect isometries between blocks of finite groups under some conditions. In this report we show those perfect isometries are isotypies. Here we recall the results in [W1] . At first we state notations from [W1]. Let S and G be finite groups such that S acts on G and $(|S|,|G|)=1$. We denote by $C_G(S)$ the subgroup of G of elements fixed by S. For a subgroup X of G if all elements of X is fixed by S, we say S centralizes X. Let $(K, \mathbf{O}, F)$ be a p-modular system such that K is algebraically closed, where p is a prime. As usual we denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G and by $\mathrm{Irr}_S(G)$ the set of S-invariant irreducible characters of G. Moreover we denote by $\pi(G, S)$ the Glauberman map from $\mathrm{Irr}_S(G)$ onto $\mathrm{Irr}(C_G(S))$ when S is solvable. Let $\mathrm{Bl}(G)$ be the set of (p-)blocks of G and $\mathrm{Bl}_S(G)$ be the set of S-invariant blocks of G. We mean a block of G a block ideal of $\mathbf{O}G$. For a block B of G we denote by $\mathrm{Irr}(B)$ the set of irreducible characters of G belonging to B and $\mathcal{R}_K(G, B)$ the additive group of characters of G afforded by KB-modules. We have the followings in [W1].

Proposition 1. Let B be an S-invarinat block of G. If S centralizes a defect group D of B, then any $\chi \in \mathrm{Irr}(B)$ is S-invarinat.

Proposition 2. Suppose that S is solvable. Let B be an S-invariant block of G such that S centralizes a defect group D of B. Then the folowings hold.

(i) Let b be a block of $C_G(S)$ containing $\pi(G, S)(\chi_1)$ for some $\chi_1 \in \mathrm{Irr}(B)$. Then there exists a perfect isometry R from $\mathcal{R}_K(G, B)$ onto $\mathcal{R}_K(C_G(S), b)$ such that $R(\chi) = \pm\pi(G, S)(\chi)$ for any $\chi \in \mathrm{Irr}(B)$.

(ii) In the situation of (i), D is a defect group of b.

Theorem 3. Suppose that S is solvable and S centralizes a Sylow p-subgroup P of G. Then there exists a unique bijection $\rho(G, S)$ from $\mathrm{Bl}_S(G)$ onto $\mathrm{Bl}(C_G(S))$ such that if $B \in \mathrm{Bl}_S(G)$ corresponds to $b \in \mathrm{Bl}(C_G(S))$ by $\rho(G, S)$, then there exists a perfect isometry R : $\mathcal{R}_K(G, B) \to \mathcal{R}_K(C_G(S), b)$ such that $R(\chi) = \pm\pi(G, S)(\chi)$ for any $\chi \in \mathrm{Irr}(B)$.

§ 2. Isotypy

At first we state the definition of the isotypy between blocks. Let B be a block of G with defect group D and (D, B_D) be a maximal B-subpair of G . We denote by $\mathbf{Br}_B(G)$ the Brauer category of B. $\mathbf{Br}_B(G)$ is the category whose objects are B-subpairs of G and whose morphisms are defined in the following way : For B-subpairs (Q, b) and (R, b') $\mathrm{Mor}((Q, b), (R, b')$)is the set of all cosets $C_G(R)g$ of G such that ${}^g(Q, b) \subseteq (R, b')$ (see [B-O], §1). We denote by $\mathbf{Br}_{B,D}(G)$ the full subcategory of $\mathbf{Br}_B(G)$ whose objects are the B-subpairs (Q, b) such that $(Q, b) \subseteq (D, B_D)$. We note that for any $Q \leq D$ there exists a unique block b such that $(Q, b) \subseteq (D, B_D)$, and we set $b = B_Q$.

Let $\mathrm{CF}(G, K)$ be the K-vector space of K-valued class functions on G and let $\mathrm{CF}(G, B, K)$ be the subspace of $\mathrm{CF}(G, K)$ of class functions α such that α is a K-linear combination of χ's in $\mathrm{Irr}(B)$. Let $\mathrm{CF}_{p'}(G, B, K)$ be the subspace of $\mathrm{CF}(G, B, K)$ of class functions vanishing on the p-singular elements of G. Let $(x, \mathbf{b})$ be a B-Brauer element of G. The decomposition map

$$d_G^{(x,\mathbf{b})} : \mathrm{CF}(G, B, K) \to \mathrm{CF}_{p'}(C_G(x), \mathbf{b}, K)$$

is defined by $d_G^{(x,\mathbf{b})}(\alpha)(y) = \alpha(xye_{\mathbf{b}})$ for any p'-element of $C_G(x)$, where $e_{\mathbf{b}}$ is the block idempotents of $\mathbf{O}C_G(x)$ corresponding to $\mathbf{b}$.

Let H be a second finite group and B' be a block of H with D as a defect group. Let (D, B'_D) be a maximal B'-subpair of H and for any subgroup Q of D let (Q, B'_Q) be the B'-subpair of H such that $(Q, B'_Q) \subseteq (D, B'_D)$.

Definition ([B], 4.6) With the above notations (G, B) and (H, B') are *isotypic* if the following conditions hold :

(i) The inclusion of D into G and H induces an equivalence of the Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{B',D}(H)$.

(ii) There exists a family of perfect isometries

$$\{R^Q : \mathcal{R}_K(C_G(Q), B_Q) \to \mathcal{R}_K(C_H(Q), B'_Q)\}_{\{Q(cyclic) \leq D\}}$$

such that for any $x \in D$

$$d_H^{(x,B'_{\langle x \rangle})} \circ R^{\langle 1 \rangle} = R_{p'}^{\langle x \rangle} \circ d_G^{(x,B_{\langle x \rangle})},$$

where $R_{p'}^{\langle x \rangle}$ is the K-linear map from $\mathrm{CF}_{p'}(C_G(x), B_{\langle x \rangle}, K)$ onto $\mathrm{CF}_{p'}(C_H(x), B'_{\langle x \rangle}, K)$ induced by $R^{\langle x \rangle}$ and we regard $R^{\langle 1 \rangle}$ as a K-linear map from $\mathrm{CF}(G, B, K)$ onto $\mathrm{CF}(H, B', K)$. In the above $R^{\langle 1 \rangle}$ is called an *isotypy* between B and B' and $(R^Q)_{\{Q(cyclic) \leq D\}}$ is called the *local system*.

Now we go back to the Glauberman correspondence situation with the condition that S centralizes a Sylow p-subgroup P of G. Then we can show that for any subgroup Q of P, S stabilizes the centralizer $C_G(Q)$ and S centralizes a Sylow p-subgroup of $C_G(Q)$ by the Schur-Zassenhaus theorem. In particular we can apply Theorem 3 for $C_G(Q)$. Let B be an S-invariant block of G with defect group D with $D \subseteq P$ and (D, B_D) be a maximal B-subpair of G. We put $b = \rho(G, S)(B)$. By Theorem 3, there exists a perfect isometry R between B and b. For any subgroup Q of D let (Q, B_Q) be a B-subpair of G contained in (D, B_D) and $b_Q = \rho(C_G(Q), S)(B_Q)$. There exists a perfect isometry R^Q between B_Q and b_Q too. We have the followings.

Proposition 4. With the just above notation, we have the following.
(i) (D, b_D) is a maximal b-subpair and (Q, b_Q) is a b-subpair contained in (D, b_D). In particular $\{ (Q, b_Q) \mid Q \leq D \}$ is the set of b-subpairs contained in (D, b_D).
(ii) The inclusion of D into G and H induces an equivalence of the Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{b,D}(C_G(S))$.

Theorem 5. With the above notation, B and b are isotypic with isotypy R and the local system $(\epsilon_Q R^Q)_{\{Q(cyclic) \leq D\}}$, where $\epsilon_Q = \pm 1$.

The proofs of the above results are reduced to the case that S is cyclic by properties of the Glauberman correspondences. Let $S = \langle s \rangle$, $\Gamma = SG$ and $C = C_G(S)$. Then we can show that there exists a block $\hat{B}$ of Γ such that $\hat{B}$ covers B, $\hat{B}$ and B are isomorphic and that for any $\chi \in \mathrm{Irr}(B)$, the extension $\hat{\chi} \in \mathrm{Irr}(\hat{B})$ of χ satisfies the following

$$\hat{\chi}(s^i c) = \delta_\chi \pi(G, S)\chi(c) \quad (\forall c \in C \textit{ and } (i, \ \mid S \mid) = 1), \quad \delta_\chi = \pm 1.$$

We call $\hat{B}$ a canonical extension of B. In general for $y \in G$ let y^G denote the class sum in $\mathbf{O}G$ of the conjugacy class of G containing y. Moreover ω_B be a linear character of the center of $\mathbf{O}G$ corresponding to B. We have the following for any $x \in C_G(S)$

$$\omega_{\hat{B}}((s^i x)^\Gamma) \equiv \omega_{\hat{B}}((s^i)^\Gamma)\omega_b(x^C) \quad (mod\ \wp) \quad (\forall x \in C \textit{ and } (i, \mid S \mid) = 1),$$

where $\wp$ is a maximal ideal of $\mathbf{O}$. The canonical extensions of S-invarinat blocks and the above congruences play a big role in our proofs of Proposition 4 and Theorem 5. For the detail, see [W2] which will appear elsewhere.

References

[B-O] M. Broué and J.B. Olsson : Subpairs multiplicities in finite grops, J. reine angew. Math. **371**(1986), 125-143.

[B] M. Broué : Isométries parfaites, types de blocs, catégories dérivées, *Astérisque* **181-182**(1990), 61-92.

[W1] A. Watanabe : Perfect isometries and the Glauberman correspondence, Proceedings of the 14th Algebraic Combinatorics Symposium, Mitaka, Tokyo, (1997), 227-230.

[W2] A. Watanabe : The Glauberman character correspondence and perfect isometries for blocks of finite groups, preprint.